Technical Appendix (online only) to
“Property Rights and Efficiency in OLG Models with Endogenous Fertility”

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July 2013

Abstract
In these notes, we provide very detailed proofs and algebra of some of the results in the Supplementary Appendix to the paper.

Contents

T.1 Detailed proof of Proposition S.1  T-1
T.2 Detailed proof of Proposition S.2  T-6
T.3 Algebra details on the equivalence between equations (S.21) to (S.23) and equations (8) to (11)  T-9
T.4 Detailed proof that $U_p^*$ is differentiable (Lemma S.4)  T-12
  T.4.1 $U_p^*(K_p, N_p^m, N_p^o; M_p)$ differentiable in $K_p > 0$  T-12
  T.4.2 $U_p^*(K_p, N_p^m, N_p^o; M_p)$ differentiable in $N_p^o > 0$  T-13
  T.4.3 $U_p^*(K_p, N_p^m, N_p^o; M_p)$ differentiable in $N_p^m > 0$  T-13
T.5 Detailed proof of Proposition S.3  T-16
T.1 Detailed proof of Proposition S.1

Proposition:
There is a unique function $U : \tilde{b}(.) \times \mathbb{X} \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \rightarrow \mathbb{R}$ satisfying equation (4).

Proof.

Step 1: If Assumptions S.2 and the boundedness condition in Assumption S.3 are satisfied and if $\kappa^\nu \zeta < 1$, there exists a unique function $U^0_D \in U_D$ that solves equation (S.10).

Proof of Claim: Assumptions S.2 and S.3 and $\kappa^\nu \zeta < 0$ ensure that we can apply Lemma 5 in Alvarez (1994) (p.37).

- That is, given $\mathcal{U} \in U_D$, define the operator $\mathcal{J} \mathcal{U}$ by

$$\mathcal{J}(\mathcal{U})(.) \equiv \beta u_D(r_i S_t - \tilde{B}(.) w_t, N^m_t) + \Psi_D(N^m_t, u_D(w_t(N^m_t + \tilde{B}(.) - S_{t+1} - \theta_i N^m_{t+1}, N^m_t) + \mathcal{U}(\tilde{x}_{Dt+1}, w_{t+1}, \tilde{y}_{t+1}).$$

- $U_D$ is complete, i.e. given $\{\mathcal{U}_i\}$, a Cauchy sequence with $\mathcal{U}_i \in U_D$, $\exists \mathcal{U} \in U_D$ such that $\nu\|\mathcal{U}_i - \mathcal{U}\| \rightarrow 0$ as $i \rightarrow \infty$.

To see this, first define $\mathcal{U}$, then show that $\mathcal{U} \in U_D$.

- Let $\mathcal{U} \equiv \lim\mathcal{U}_i$ pointwise, which is well-defined since $\{\mathcal{U}_i\}$ is Cauchy.

Since $\mathcal{U}_i(\tilde{x}_D) \rightarrow \mathcal{U}(\tilde{x}_D)$, $\forall \tilde{x}_D \in \Pi_D$, given $\delta$ choose $i$ big enough so that

$$\forall \tilde{x}_D \in \Pi_D, |\mathcal{U}_i(\tilde{x}_D) - \mathcal{U}(\tilde{x}_D)|/||\tilde{x}_D||^\nu \leq \delta.$$

Hence, $\nu\|\mathcal{U}_i - \mathcal{U}\| = \sup\{|\mathcal{U}_i(\tilde{x}_D) - \mathcal{U}(\tilde{x}_D)|/||\tilde{x}_D||^\nu, \tilde{x}_D \in \Pi_D\} \rightarrow 0$ as $i \rightarrow \infty$.

- $\mathcal{U} \in U_D$:

* $\nu\|\mathcal{U}\| < \infty$:

  given $\delta$, choose $i$ large enough so that

  $$\forall \tilde{x}_D \in \Pi_D, |\mathcal{U}_i(\tilde{x}_D)| \leq |\mathcal{U}_i(\tilde{x}_D)| + \delta ||\tilde{x}_D||^\nu.$$

  Since $\mathcal{U}_i \in U_D$, let $B_i \equiv \nu\|\mathcal{U}_i\| < \infty$ and note that

  $$\forall \tilde{x}_D \in \Pi_D, |\mathcal{U}_i(\tilde{x}_D)| \leq B_i ||\tilde{x}_D||^\nu.$$

  Hence, for $i$ large enough,

  $$\forall \tilde{x}_D \in \Pi_D, |\mathcal{U}(\tilde{x}_D)| \leq (B_i + \delta) ||\tilde{x}_D||^\nu.$$

  Hence, $\nu\|\mathcal{U}\| = \sup\{|\mathcal{U}(\tilde{x}_D)|/||\tilde{x}_D||^\nu\} \leq (B_i + \delta) \rightarrow \infty$

* $\mathcal{U}$ continuous at $\tilde{x}_D \in \Pi_D$:

  need to show that for any $\xi > 0$, $\exists \delta > 0$ such that if $||\tilde{x}_D - y_D||^\nu \leq \delta$, then $|\mathcal{U}(\tilde{x}_D) - \mathcal{U}(y_D)| \leq \xi.$
Given $\xi$ and $\varphi_{D}$, choose $i$ big enough such that
\[ \nu||U_{i} - U|| \leq (\xi/4)||\varphi_{D}||_{\nu}, \]
and choose $\delta$ such that
\[ \text{if } ||\varphi_{D} - y_{D}||_{\nu} \leq \delta, \text{ then } |U_{i}(\varphi_{D}) - U_{i}(y_{D})| < \xi/4 \text{ and } (||y_{D}||_{\nu}/||\varphi_{D}||_{\nu}) \leq 2. \]

Hence, we have
\[ |U(\varphi_{D}) - U(y_{D})| \leq |U(\varphi_{D}) - U_{i}(\varphi_{D})| + |U_{i}(\varphi_{D}) - U_{i}(y_{D})| + |U_{i}(y_{D}) - U(y_{D})| \]
\[ \leq \nu||U_{i} - U|| ||\varphi_{D}||_{\nu} + |U_{i}(\varphi_{D}) - U_{i}(y_{D})| + \nu||U_{i} - U|| ||y_{D}||_{\nu} \]
\[ \leq \xi/4 + \xi/4 + (\xi/4)(||y_{D}||_{\nu}/||\varphi_{D}||_{\nu}) \leq \xi/4 + \xi/4 + \xi/2 = \xi. \]

\[ * \text{ } U \text{ is h.o.d. } \nu: \]
\[ |U(\varphi_{D}) - \lambda^{\nu}U(\varphi_{D})| \leq |U(\varphi_{D}) - U_{i}(\varphi_{D})| + \lambda^{\nu}|U_{i}(\varphi_{D}) - U(\varphi_{D})|. \]
For any $\xi$, choose $i$ big enough so that
\[ |U(\varphi_{D}) - U_{i}(\varphi_{D})| \leq \xi/2 \text{ and } |U_{i}(\varphi_{D}) - U(\varphi_{D})| \leq (\xi/2)/\lambda^{\nu} \]
Hence, $|U(\varphi_{D}) - \lambda^{\nu}U(\varphi_{D})| \leq \xi$. Since $\xi$ was arbitrary, $U(\varphi_{D}) = \lambda^{\nu}U(\varphi_{D})$.

Hence, $U_{D} \in U_{D}$.

\[ \bullet \text{ } \mathcal{J} : U_{D} \rightarrow U_{D}: \]

- $(\mathcal{J}U)(.)$ is continuous because $u_{D}$ and $\Psi_{D}$ are continuous;
- $(\mathcal{J}U)(.)$ is h.o.d. $\nu$:
  \[ (\mathcal{J}U)(\lambda_{D}t) = u_{D}(\tilde{C}^{o}(\lambda_{D}t, \lambda_{D}t+1), \lambda_{D}t) \]
  \[ + \Psi_{D}[N_{D}^{m}, u_{D}(\tilde{C}^{m}(\lambda_{D}t, \lambda_{D}t+1), N_{D}^{m}) + U(\lambda_{D}t+1)]. \]

which can be done because the constraint in the definition of $\Pi(x_{D0}, w_{D}, r_{D})$ is h.o.d. 1.

Since $C_{D}$ h.o.d. 1 in $(x_{D1}, x_{D1+1})$, we get
\[ (\mathcal{J}U)(\lambda_{D}t) = u_{D}(\tilde{C}^{o}(x_{D1}, x_{D1+1}), \lambda_{D}t) \]
\[ + \Psi_{D}[N_{D}^{m}, u_{D}(\tilde{C}^{m}(x_{D1}, x_{D1+1}), N_{D}^{m}) + U(x_{D1+1})]. \]

Since $u_{D}$ and $U$ are h.o.d. $\nu$ and $\Psi_{D}$ is h.o.d. $\nu$ in the sense of Assumptions S.2.
\[ (\mathcal{J}U)(\lambda_{D}t) = \lambda^{\nu}(u_{D}(\tilde{C}^{o}(x_{D1}, x_{D1+1}), N_{D}^{o})) \]
\[ + \Psi_{D}[N_{D}^{m}, u_{D}(\tilde{C}^{m}(x_{D1}, x_{D1+1}), N_{D}^{m}) + U(x_{D1+1})]] \]
\[ (\mathcal{J}U)(\lambda_{D}t) = \lambda^{\nu}(\mathcal{J}U)(\varphi_{D}). \]
- \( \nu \|\mathcal{J}U\| < \infty \):

By Assumption S.3, \( \exists B \) such that

\[
\forall y \in \Pi_D(y), |\mathcal{J}U(y_D)| \leq B\{(||y_D||_E)^\nu + (||y_D+1||_E)^\nu + |U(y_D+1)|\}
\]

By definition of \( \kappa, \kappa''(||y_D||_E)^\nu \geq (||y_D+1||_E)^\nu \). Hence, we have

\[
\forall y \in \Pi_D(y), |\mathcal{J}U(y_D)| \leq B\{(||y_D||_E)^\nu(1 + \kappa'') + |U(y_D+1)|\}
\]

Let \( y_{D0} = y_{D1}/\kappa' \), i.e. \( y_{D1} = \kappa' x_D \). Then, since \( \mathcal{J}U(.) \) h.o.d. \( \nu \) and \( \Pi \) h.o.d. 1, we have

\[
\forall y_{D0} \in \Pi_D(x_0), (\kappa'')^t|\mathcal{J}U(y_{D0})| \leq B\{(\kappa'')^t(||x_{D0}||_E)^\nu(1 + \kappa'') + (\kappa'')^{t+1}U(y_{D0})\}.
\]

Divide both sides by \( (\kappa'')^t||x_{D0}||_E^{\kappa'} \) to get

\[
\forall y_{D0} \in \Pi_D(x_0), \frac{|\mathcal{J}U(x_{D0})|}{||x_{D0}||_E^{\kappa'}} \leq B\{(||x_{D0}||_E)^\nu(1 + \kappa'') + (\kappa'')^{t+1}U(y_{D0})\}.
\]

Since, \( \kappa > 0 \) implies \( ||x_{D0}||_E = ||x_{D0}||_E \), we get

\[
\forall y_{D0} \in \Pi_D(x_0), \frac{|\mathcal{J}U(x_{D0})|}{||x_{D0}||_E^{\kappa'}} \leq B\{(||x_{D0}||_E)^\nu(1 + \kappa'') + (\kappa'')^{t+1}U(x_{D0})\}.
\]

Since \( U \in \mathcal{U}_D \), \( |U(y_{D0})|/||x_{D0}||_E < \infty \).

Hence, \( \forall x_{D0} \in X_D, \forall y_{D0} \in \Pi_D(x_{D0}), |\mathcal{J}U(y_{D0})|/||x_{D0}||_E^{\kappa'} < \infty \) and

\[\nu |\mathcal{J}U| = \sup\{|\mathcal{J}U(y_{D0})|/||x_{D0}||_E^{\kappa'} : y_{D0} \in X_D\} < \infty.\]

- \( \mathcal{J} \) is a contraction of modulus \( \zeta \).

Given \( a > 0 \) and \( U \in \mathcal{U}_D \), define \( (U + a)(x_D) \equiv U(x_D) + a||x_D||_E^{\kappa'} \). Note that \( (U + a)(.) \in \mathcal{U}_D \).

**Lemma T.1** (Modified Blackwell) Let \( M : \mathcal{U}_D \to \mathcal{U}_D \), let \( U, G \in \mathcal{U}_D \).

If (i) \( M \) is monotone, i.e. \( U \geq G \Rightarrow MU \geq MG \) and

if (ii) \( M \) discounts, i.e. \( M(U + a) \leq MU + \theta a \),

then \( M \) is a contraction of modulus \( \theta \).

**Proof.** The proof is standard except for the definition of \( (U + a)(.) \). See Alvarez (1994) (Proposition 3). \[\tag*{\Box}\]

- \( \mathcal{J} \) is monotone: follows immediately from Assumption S.2(b).

- \( \mathcal{J} \) discounts:

\[
\mathcal{J}(U + a)(x_{D1}) = u_D(\tilde{C}^0(x_{D1}, x_{D1+1}), N^o) + \Psi_D[N^o, u_D(\tilde{C}^m(x_{D1}, x_{D1+1}), N^m) + (U + a)(x_{D1+1})]
\]

By the definition of \( (U + a)(.) \),

T-3
\[
\mathcal{J}(\mathcal{U} + a)(\tilde{x}_{Dt}) = u_{D}(\widetilde{C}^o(\tilde{x}_{Dt}, x_{Dt+1}), N_t^o)
\]
\[
+ \Psi_D(N_t^m, u_{D}(\tilde{C}^m(x_{Dt}, x_{Dt+1}), N_t^m) + U(\tilde{x}_{Dt}) + a\|\tilde{x}_{Dt+1}\|_E \]
\]

Since, \( \kappa > 0 \) implies \( \|\tilde{x}_{Dt}\|_E = \|x_{Dt}\|_E \) and since \( \Psi_D \) discounts, we get
\[
\mathcal{J}(\mathcal{U} + a)(\tilde{x}_{Dt}) \leq u_{D}(\widetilde{C}^o(\tilde{x}_{Dt}, x_{Dt+1}), N_t^o)
\]
\[
+ \Psi_D(N_t^m, u_{D}(\tilde{C}^m(x_{Dt}, x_{Dt+1}), N_t^m) + U(\tilde{x}_{Dt}) + a\|\tilde{x}_{Dt}\|_E \]
\]
\[
\mathcal{J}(\mathcal{U} + a)(\tilde{x}_{Dt}) \leq (\mathcal{J}\mathcal{U})(\tilde{x}_{Dt}) + \zeta a\|x_{Dt}\|_E = (\mathcal{J}\mathcal{U})(\tilde{x}_{Dt}) + \zeta a\|x_{Dt}\|_E^o
\]
\[
= (\mathcal{J}\mathcal{U} + \zeta a)(\tilde{x}_{Dt})
\]

Hence, \( \mathcal{J} \) discounts.

Hence, \( \mathcal{J} \) is a contraction of modulus \( \zeta \).

Hence, \( \mathcal{J} \) has a unique fixed point, \( U_D^o = \mathcal{J}U_D^o \).

**Step 2:** Given \( U_D^o \), for all \( x_0 \in \mathbb{R}^2_+ \), \( \bar{x}_0 \in \Pi(x_0, \psi_0, \lambda_0) \), define
\[
U^o(\bar{x}_t; \psi_t, \lambda_t) = \frac{U_D^o(\tilde{x}_{Dt}; \psi_t, \lambda_t)}{(N_t^o)^\nu}
\]

where \( x_{Dt} = N_t^o(x_t, 1) \).

- **Claim i:** \( U^o \) solves equation (S.3).

**Proof of Claim i:**
\[
U^o(\bar{x}_t; \psi_t, \lambda_t) = \frac{U_D^o(\tilde{x}_{Dt}; \psi_t, \lambda_t)}{(N_t^o)^\nu}
\]
\[
= \frac{\beta u_{D}(r_tS_t - \tilde{B}(.)\psi_t, N_t^m)}{(N_t^o)^\nu}
\]
\[
+ \Psi_D(N_t^m, u_{D}(\psi_t(N_t^m + \tilde{B}(.) - S_{t+1} - \theta_tN_{t+1}^m) + U_D^o(\tilde{x}_{Dt+1}; \psi_{t+1}, \lambda_{t+1}))\]
\]

By definition of \( U^o \) and \( u_{D,}\),
\[
= \frac{\beta(N_t^o)^\nu u((r_tS_t - \tilde{B}(.)\psi_t)/N_t^o)}{(N_t^o)^\nu}
\]
\[
+ \Psi_D(N_t^m, (N_t^o)^\nu u((\psi_t(N_t^m + \tilde{B}(.) - S_{t+1} - \theta_tN_{t+1}^m)/N_t^m) + (N_t^o)^\nu U^o(\tilde{x}_{t+1}; \psi_{t+1}, \lambda_{t+1})))
\]

T-4
Since $N_{t+1}^o = N_{t}^m = N_{t}^o \frac{N_{t}^m}{N_{t}^o} = N_{t}^o n_{t-1}$, $\Psi_D$ h.o.d. $\nu$ and by definition of $\Psi_D$, $S_t$ and $B_t$

\[
(N_{t}^o)^\nu \left[ \frac{\beta u([r_t S_t - \tilde{B}(.) w_t]/N_{t}^o)}{(N_{t}^o)^\nu} 
\right. 
+ \left. \frac{\Psi_D(n_{t-1}, (n_{t-1})^\nu \{u([w_t(N_t^m + \tilde{B}(.)) - S_{t+1} - \theta_t N_{t+1}^m]/N_t^m) + U^o(x_{t+1}; w_{t+1}, r_{t+1})\})}{(N_{t}^o)^\nu} \right]
\]

\[
= \beta u([r_t S_t - \tilde{B}(.) w_t]/N_{t}^o)
+ \Psi(n_{t-1}, u([w_t(N_t^m + \tilde{B}(.)) - S_{t+1} - \theta_t N_{t+1}^m]/N_t^m) + U^o(x_{t+1}; w_{t+1}, r_{t+1}))
= \beta u(r_t S_t - w_t n_{t-1} b(.) + \Psi(n_{t-1}, u(w_t (1 + b(.)) - s_{t+1} - \theta n_t) + U^o(x_{t+1}; w_{t+1}, r_{t+1})).
\]

Hence, $U^o$ solves equation (S.3).

- **Claim ii:** $U^o$ is unique.

**Proof of Claim ii:** Suppose there was another function, $\tilde{U}^o$ that solves equation (S.3). Then $\tilde{U}^o_D = (N^o)^\nu \tilde{U}^o$ would be different from $U^o_D$ but solve equation (S.10): a contradiction since $U^o_D$ is unique.

**Step 3:** Given $U^o$, for all $x_0 \in \mathbb{R}^2_+$, $x_0 \in \Pi(x_0, w_0, r_0)$ and for $b_t = \tilde{b}(x_t, x_{t+1}; w_t, r_t)$, define

\[
U^m(b_t, x_t; w_t, r_t) = u(w_t (1 + b_t) - s_{t+1} - \theta n_t) + U^o(x_{t+1}; w_{t+1}, r_{t+1}).
\]

- **Claim iii:** $U^m$ solves equation (4).

**Proof of Claim iii:**

Using equation (S.3),

\[
= u(w_t (1 + b_t) - s_{t+1} - \theta n_t) + \beta u(r_{t+1} s_{t+1} - w_{t+1} n_t b(.) + \Psi(n_t, u(w_{t+1} (1 + b(.) - s_{t+2} - \theta n_{t+1}) + U^o(x_{t+2}; w_{t+2}, r_{t+2})).
\]

Since $b_{t+1} = b(x_{t+1}, x_{t+2}; w_{t+1}, w_{t+2}),$

\[
= u(w_t (1 + b_t) - s_{t+1} - \theta n_t) + \beta u(r_{t+1} s_{t+1} - w_{t+1} n_t b_{t+1}) + \Psi(n_t, u(w_{t+1} (1 + b_{t+1}) - s_{t+2} - \theta n_{t+1}) + U^o(x_{t+2}; w_{t+2}, r_{t+2})).
\]
By definition of $U^m$,

$$= u(w_t(1 + b_t) - s_{t+1} - \theta_n_t) + \beta u(r_{t+1}s_{t+1} - w_{t+1}n_{t+1}) + \Psi(n_t, U^m(b_{t+1}, \xi_{t+1}; \lambda_{t+1}); \xi_{t+1}).$$

Hence, $U^m$ solves equation (4).

- **Claim iv**: $U^m$ is unique.

**Proof of Claim iv**: Suppose there was another function, $\tilde{U}^m$ that solves equation (4).

Then $\tilde{U}^m_{t+1} \equiv \tilde{U}_t^m - u(w_t(1 + b_t) - s_{t+1} - \theta n_t)$ would be different from $U^o$ but solve equation (S.3): a contradiction since $U^o$ is unique.

\[\blacksquare\]

### T.2 Detailed proof of Proposition S.2

In this section we provide the details for the proof of Proposition S.2. For necessity of equations (S.22) and (S.23), we closely follow the arguments in Alvarez (1994), Prop. 8. For sufficiency of equations (S.22) and (S.23), we closely follow the proof of Theorem 4.15 in Stokey, Lucas, and Prescott (1989), p. 98.

**Remark 2** Note that Step 1 in Proposition S.1 can be used to show that $U^o_P$ is uniquely defined through equation (S.19).

**Proposition:**

If $M_P = F_N$, any sequence $x_{P0}^o \in \Pi_P(x_{P0})$ is optimal for the problem in Definition S.4 if and only if it satisfies the intra-temporal allocation condition (S.21), the Euler equations (S.22) and the transversality condition (S.23).

**Proof. Necessity:** Suppose an allocation attains the sup in Definition S.4, $U^o_P(x_{P0})$ where $U^o_P = U^*_P$ by Lemma S.3 and where $\tilde{Q}(\cdot)$ is defined in equation (S.17). Then, since $U^o_P$ is strictly increasing, concave and differentiable by Lemma S.4, the following conditions hold necessarily:

\[
\begin{align*}
B : \quad & \beta \frac{\partial u_D(C^o, N^o)}{\partial C^o} = \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U'_P) \partial u_D(C^m, N^m)}{\partial C^m} + \lambda_B/M_P \\
K' : \quad & \frac{\partial u_D(C^m, N^m)}{\partial C^m} = \frac{\partial U'_P}{\partial K'} \\
N^{m'} : \quad & \gamma \frac{\partial u_D(C^m, N^m)}{\partial C^m} = \frac{\partial U'_P}{\partial N^{m'}}
\end{align*}
\]
where the first equation is equation (S.20) for \( z = U'_P \).

The envelope conditions for the Pseudo-Planner are given by

\[
\begin{align*}
K : & \quad \frac{\partial U'_P}{\partial K} = \beta \frac{\partial u_D(C^o, N^o)}{\partial C^o} F_K \\
N^o : & \quad \frac{\partial U'_P}{\partial N^o} = \beta \frac{\partial u_D(C^o, N^o)}{\partial N^o} \\
N^m : & \quad \frac{\partial U'_P}{\partial N^m} = \frac{\partial u_D(C^o, N^o)}{\partial C^o} (F_N - M_P) + \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U'_P)}{\partial N^m} \\
& \quad \quad \quad \quad + \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U'_P)}{\partial C^m} \left[ M_P \frac{\partial u_D(C^m, N^m)}{\partial C^m} + \frac{\partial u_D(C^m, N^m)}{\partial N^m} + \frac{\partial U'_P}{\partial N^o} \right] \\
& \quad \quad \quad \quad - \left( \frac{\beta \partial u_D(C^o, N^o)}{\partial C^o} - \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U'_P)}{\partial U} \frac{\partial u_D(C^m, N^m)}{\partial C^m} \right) bM_P.
\end{align*}
\]

Substituting out \( \frac{\partial U'_P}{\partial x} \) and setting \( M_P = F_N \), we get the conditions in (S.21) and (S.22).

To show that the transversality condition must hold at an optimum, consider the following claim.

**Claim 3** For any feasible sequence, \( \bar{x}_{P_t} \), that follows \( x_{P_0} \), we have \( \zeta^t U'_P(\bar{x}_{P_t}) \to 0 \) as \( t \to \infty \).

**Proof of Claim:** Using Assumption S.2(e) (i.e. Assumption S.3), using \( ||x_{P_t}||_E \leq (k')^t ||x_{P_0}||_E \) (by Remark 1), the fact that \( \kappa > 0 \) implies \( ||\bar{x}_{P_t}||_{\kappa} = ||x_{P_0}||_E \) knowing that \( \nu ||U'_P|| < \infty \) (by Remark 2 and Proposition S.1), multiplying by \( \zeta^t \) and using \( \zeta \kappa^t < 1 \), the result follows.

In particular, \( \zeta^t U'_{P^o}(x_{P_t}) \to 0 \) for any feasible \( x_{P_t} \) starting from \( x_{P_0} \). But \( U'_{P^o}(x_{P_t}) = U'_{P}(x_{P_t}) \). Since \( U'_{P^o}(x_{P_t}) \) is hod \( \nu \), we have:

\[
\nu U'_{P}(x_{P_t}) = \frac{\partial U'_{P^o}(x_{P_t})}{\partial K_t} K_t + \frac{\partial U'_{P^o}(x_{P_t})}{\partial N^m_t} N^m_t + \frac{\partial U'_{P^o}(x_{P_t})}{\partial N^o_t} N^o_t.
\]

Hence,

\[
\zeta^t \left[ \frac{\partial U'_{P^o}(x_{P_t})}{\partial K_t} K_t + \frac{\partial U'_{P^o}(x_{P_t})}{\partial N^m_t} N^m_t + \frac{\partial U'_{P^o}(x_{P_t})}{\partial N^o_t} N^o_t \right] \to 0 \text{ as } t \to \infty.
\]

Using the envelope conditions as well as the Euler equations, we get the transversality condition in (S.23) as desired.

**Sufficiency:** Suppose \( x'_{P_0} \) satisfies equations (S.22) and (S.23). Define

\[
D_t \equiv U'_P(\bar{x}_{P_t}) - U'_P(\bar{x}_{P_t})
\]

T-7
some $x_{Pt} \in \Pi_P(x_{Pt})$. All we need to show is that $D_0 \geq 0$, $\forall x_{P0} \in \Pi_P(x_{P0})$. By concavity of $u_D(o,.)$, $\Psi_D$ and $u_D(m,.)$ we can write Again, by concavity of $u_D(m,.)$, we have

$$D_t \geq \Xi_t + \frac{\partial \Psi_D(t,\ast)}{\partial U} D_{t+1}$$

where $\Xi_t$ contains a bunch of utility derivatives evaluated at $x_{Pt}^*$ multiplied by $(x_{Pt}^* - x_{Pt})$. Sequentially using this inequality by substituting for $D_{t+1}$ from $t = 0$ (with the standard convention that $\prod_{s=0}^{\infty} \frac{\partial \Psi_D(s,\ast)}{\partial U} = 1$) gives:

$$D_0 \geq \lim_{T \to \infty} \left\{ \sum_{t=0}^{T} \left( \prod_{s=0}^{t-1} \frac{\partial \Psi_D(s,\ast)}{\partial U} \right) \Xi_t + \left( \prod_{s=0}^{T} \frac{\partial \Psi_D(s,\ast)}{\partial U} \right) D_{T+1} \right\}$$

Since the initial condition is given, we have $x_{P0}^* = x_{P0}$. Hence, several terms disappear in $t = 0$. Rearranging terms over time and grouping by time period (as opposed to age), it is straightforward to see that most of $\Xi_t$ cancels using equation (S.21). What remains is:

$$D_0 \geq -\lim_{T \to \infty} \left\{ \left( \prod_{s=0}^{T} \frac{\partial \Psi_D(s,\ast)}{\partial U} \right) \ast \left[ \frac{\partial u_D(C_{T+1}^{m,s},N_{T+1}^{m,s})}{\partial C_{T+1}^{m,s}} (K_{T+1}^* - K_{T+1}) + \theta(N_{T+1}^{m,s} - N_{T+1}^m) \right] \right\}$$

$$\geq -\lim_{T \to \infty} \left\{ \left( \prod_{s=0}^{T} \frac{\partial \Psi_D(s,\ast)}{\partial U} \right) \ast \left[ \frac{\partial u_D(C_{T+1}^{m,s},N_{T+1}^{m,s})}{\partial C_{T+1}^{m,s}} (K_{T+1}^* + \theta N_{T+1}^{m,s}) + \beta \frac{\partial u_D(C_{T+1}^{m,s},N_{T+1}^{m,s})}{\partial N_{T+1}^{m,s}} (N_{T+1}^{m,s} - N_{T+1}^m) \right] \right\}$$

$$\geq -\lim_{T \to \infty} \left\{ \left( \prod_{s=0}^{T} \frac{\partial \Psi_D(s,\ast)}{\partial U} \right) \ast \left[ \frac{\partial u_D(C_{T+1}^{m,s},N_{T+1}^{m,s})}{\partial C_{T+1}^{m,s}} (K_{T+1}^* + \theta N_{T+1}^{m,s}) + \beta \frac{\partial u_D(C_{T+1}^{m,s},N_{T+1}^{m,s})}{\partial N_{T+1}^{m,s}} (N_{T+1}^{m,s} - N_{T+1}^m) \right] \right\}$$

where the last inequality follows from the monotonicity of $\Psi_D$ and $u_D$.

Using the EE for $K$, we get:

$$D_0 \geq -\beta \lim_{T \to \infty} \left\{ \left( \prod_{s=0}^{T} \frac{\partial \Psi_D(s,\ast)}{\partial U} \right) \ast \left[ \frac{\partial u_D(C_{T+1}^{m,s},N_{T+1}^{m,s})}{\partial C_{T+1}^{m,s}} F_{K_{T+1}}(K_{T+1}^* + \theta N_{T+1}^{m,s}) + \frac{\partial u_D(C_{T+1}^{m,s},N_{T+1}^{m,s})}{\partial N_{T+1}^{m,s}} (N_{T+1}^{m,s} - N_{T+1}^m) \right] \right\}$$

This almost immediately leads to the result using Claims 3 and 4.
Claim 4  \( \frac{\partial \psi_D(N^m, U)}{\partial u} \leq \zeta < 1 \)

**Proof of Claim:** By Assumption S.2(c), one can easily show (taking derivatives wrt \( \lambda \) and setting \( \lambda = 1 \)) that:

\[
\frac{\partial \psi_D(N^m, U)}{\partial N^m} N^m + \nu \frac{\partial \psi_D(N^m, U)}{\partial N^m} U = \nu \psi_D(N, U). \tag{T.1}
\]

By Assumption S.2(d), \( \psi_D \) discounts at rate \( \zeta < 1 \). Using the homogeneity property in equation (T.1) and taking selective limits for \( a \to 0 \) in the discounting inequality, we get the desired result.

Using Claims 4 and then 3, we get that the second term converges to 0. Using Claim 4, it then follows from the transversality condition in equation (S.23) that the first term goes to 0 as well. Hence, \( D_0 \geq 0 \). \( \square \)

### T.3 Algebra details on the equivalence between equations (S.21) to (S.23) and equations (8) to (11)

Note that

\[
u_D(C, N) = N^\nu u(C/N)
\]

\[
\frac{\partial \psi_D}{\partial C} = N^{\nu-1} u'(c)
\]

\[
\frac{\partial \psi_D}{\partial N} = N^{\nu-1} (\nu u(c) - cu'(c))
\]

\[
\psi_D(N^m, u_D(C, N^m) + U_{p'}) = \psi(N^m, (u_D(C, N^m) + U_{p'})/(N^m)^\nu)
\]

\[
= \psi(N^\nu n, (N^m)^\nu(u(c^m) + U_{p'})/(N^m)^\nu)
\]

\[
= (N^\nu)^\nu \psi(n, u(c^m) + U_{p'})
\]

\[
\frac{\partial \psi_D}{\partial N^m} = \psi(n, (u_D(C, N^m) + U_{p'})/(N^m)^\nu)
\]

\[
- \psi_U(N^m, (u_D(C, N^m) + U_{p'})/(N^m)^\nu)\nu(N^m)^\nu-1(u_D(C, N^m) + U_{p'})
\]

\[
= (N^\nu)^{\nu-1} \psi(n, u(c^m) + U_{p'})
\]

\[
- (N^\nu)^{\nu-1} \psi_U(n, u(c^m) + U_{p'})\nu(N^m)^{-1}(u(c^m) + U_{p'})
\]

\[
= (N^\nu)^{\nu-1} \psi_U(n, u(c^m) + U_{p'})\nu(N^m)^{-1}(u(c^m) + U_{p'})
\]

\[
\frac{\partial \psi_D}{\partial U} = (N^m)^{-\nu} \psi_U(N^m, (u_D(C, N^m) + U_{p'})/(N^m)^\nu)
\]

\[
= (N^\nu)^{\nu-1} \psi_U(n, u(c^m) + U_{p'}) = n^{-\nu} \psi_U(n, u(c^m) + U_{p'})
\]

\[
\lambda_B = (N^m_i)^{\nu-1} n_{t-1} \lambda_b
\]

**Intra-temporal allocation condition:**
Using this, equation (S.21) becomes:

\[
\beta(N_t^u)^{v-1}u'(c_t^m) = (n_t-1)^{-\nu}\Psi_U(N_t^m)^{v-1}u'(c_t^m) + (N_t^m)^{\nu-1}\lambda_b,t n_{t-1}^{-\nu}/F_{Nt}
\]

\[
\beta n_{t-1}^{-\nu}u'(c_t^o) = (n_t-1)^{-\nu}\Psi_U u'(c_t^m) + \lambda_b,t n_{t-1}^{-\nu}/F_{Nt}
\]

\[
\beta n_{t-1}u'(c_t^o) = \Psi_U(n_{t-1}, U_t)u'(c_t^m) + \lambda_b,t/F_{Nt}
\]

Using the firm’s optimality, this boils down to equation \((8)\).

**Euler Equations:**

Equations (S.22) become:

\[
(N_t^m)^{v-1}u'(c_t^m) = \beta(N_{t+1}^o)^{\nu-1}u'(c_{t+1}^o)F_{Kt+1}
\]

\[
\theta(N_t^m)^{v-1}u'(c_t^m) = (N_t^m)^{\nu-1}(\Psi_u(n_t, U_{t+1}) - \Psi_U(n_t, U_{t+1})\nu n_{t-1}U_{t+1})
\]

\[
+ n_t^{-\nu}\Psi_U(n_t, U_{t+1})
\]

\[
* [F_{Nt+1}(N_{t+1}^m)^{v-1}u'(c_{t+1}^m) + (N_{t+1}^m)^{\nu-1}(\nu u(c_{t+1}^m) - c_{t+1}^m u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^m) - c_{t+2}^o u'(c_{t+2}^o))]
\]

\[
- \lambda_b,t+1(N_{t+1}^m)^{\nu-1}n_{t-1}^{-\nu}b_{t+1}
\]

Dividing both sides by \((N_t^m)^{\nu-1}\) in both equations gives:

\[
u'u'(c_t^m) = \beta u'(c_{t+1}^o)F_{Kt+1}
\]

\[
\theta u'(c_t^m) = \Psi_u(n_t, U_{t+1}) - n_t^{-1}\Psi_U(n_t, U_{t+1})\nu (u(c_{t+1}^o) + U_{t+2}^o)
\]

\[
+ n_t^{-1}\Psi_U(n_t, U_{t+1})[F_{Nt+1}u'(c_{t+1}^m) + (\nu u(c_{t+1}^m) - c_{t+1}^m u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^m) - c_{t+2}^o u'(c_{t+2}^o))]
\]

\[
- \lambda_b,t+1n_{t-1}^{-1}b_{t+1}
\]

The first equation boils down to equation 6. For the second, let’s cancel some terms:

\[
\theta u'(c_t^m) = \Psi_u(n_t, U_{t+1}) - n_t^{-1}\Psi_U(n_t, U_{t+1})\nu U_{t+2}^o
\]

\[
+ n_t^{-1}\Psi_U(n_t, U_{t+1})[(F_{Nt+1} - c_{t+1}^m)u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^m) - c_{t+2}^o u'(c_{t+2}^o))]
\]

\[
- \lambda_b,t+1n_{t-1}^{-1}b_{t+1}
\]
Using the budget constraint when young gives:

\[
\theta u'(c_i^m) = \Psi_n(n_t, U_{t+1}) + n_t^{-1} \Psi_U(n_t, U_{t+1}) \\
*[(s_{t+1} + \theta n_{t+1} - b_{t+1} w_{t+1})u'(c_{t+1}^m) + \beta (nu(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) - \nu U_{t+2}^o] \\
- \lambda_{b,t+1} n_t^{-1} b_{t+1}
\]

Using the equation we derived from equation (S.21)

\[
\theta u'(c_i^m) = \Psi_n(n_t, U_{t+1}) - n_t^{-1}(\beta n_t u'(c_{t+1}^o) - \lambda_{b,t+1}/F_{N_t+1})b_{t+1} w_{t+1} \\
+n_t^{-1} \Psi_U(n_t, U_{t+1})[(s_{t+1} + \theta n_{t+1})u'(c_{t+1}^m) + \beta (nu(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) - \nu U_{t+2}^o] \\
- \lambda_{b,t+1} n_t^{-1} b_{t+1}
\]

Now, clearly, the terms in \( \lambda_{b,t+1} \) cancel out because either the constraint is not binding and \( \lambda_{b,t+1} = 0 \) or it is binding and \( b_{t+1} = \frac{\lambda}{n_t} \).

\[
\theta u'(c_i^m) = \Psi_n(n_t, U_{t+1}) - n_t^{-1}(\beta n_t u'(c_{t+1}^o) - \lambda_{b,t+1}/F_{N_t+1})b_{t+1} w_{t+1} \\
+n_t^{-1} \Psi_U(n_t, U_{t+1})[(s_{t+1} + \theta n_{t+1})u'(c_{t+1}^m) + \beta (nu(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) - \nu U_{t+2}^o]
\]

We know, from the envelope condition, that

1. \( \partial U_{PT+2}^o / \partial N_t^o = \beta \partial u_D(o, t+2) / \partial N_t^o = (N_t^o)^{\nu-1}\beta (nu(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) \)

and, from the homogeneity of \( U_P^o \), that

2. \( \partial U_{PT+2}^o / \partial N_t^o = \nu U_{PT+2}^o / N_t^o - (\partial U_{PT+2}^o / \partial K_{t+2}) K_{t+2} / N_t^o - (\partial U_{PT+2}^o / \partial N_t^o) N_{t+2}^o / N_t^o \)

Or, using the FOCs for \( K \) and \( N \),

2. \( \partial U_{PT+2}^o / \partial N_t^o = \nu U_{PT+2}^o / N_t^o - \left( \frac{\partial u_D(c_{t+1}^m, N_{t+1}^m)}{\partial c_{t+1}^m} \right) s_{t+2} - \left( \frac{\partial u_D(c_{t+1}^m, N_{t+1}^m)}{\partial c_{t+1}^m} \right) \theta n_{t+1} \\
= (N_t^o)^{\nu-1}[\nu U_{t+2}^o - u'(c_{t+1}^m)(s_{t+2} + \theta n_{t+1})]
\]

Hence, \( \beta (nu(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) = \nu U_{t+2}^o - u'(c_{t+1}^m)(s_{t+2} + \theta n_{t+1}) \).

That is, the second term on the RHS cancels out and we get equation (7).

Transversality condition:

Using the relationships in the beginning of the section and dividing by \( N_t^o \), equation (S.23) becomes:

\[
\zeta^t (N_t^o)^{\nu-1}[u'(c_i^o)F_{Kt}(K_t + \theta N_t^m) + (nu(c_i^o) - c_i^o u'(c_i^o))N_t^o] \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty
\]

T-11
Dividing by $(N_t^o)^v$, 

$$\zeta^t[u'(c_t^o)F_{Kt}(s_t + \theta n_{t-1}) + \nu u(c_t^o) - c_t^o u'(c_t^o)] \to 0$$

Using the budget constraint when old:

$$\zeta^t[u'((F_{Kt}k_t - b_t F_N t)n_{t-1})(F_{Kt}\theta - b_t F_{Nt})n_{t-1}) + \nu u((F_{Kt}k_t - b_t F_{Nt})n_{t-1})] \to 0$$

That is, we get equation (11).

### T.4 Detailed proof that $U_P^o$ is differentiable (Lemma S.4)

#### T.4.1 $U_P^o(K_P, N_P^m, N_P^o; M_P)$ differentiable in $K_P > 0$

Define

$$G(A) \equiv \max_{C^m, C^o, B} \{\beta U(C^o, N_P^o) + \Psi_D[N_P^m, U(C^m, N_P^m) + U_P^o(x_P^*(x_P; M_P); M_P)]\}$$

subject to

$$C^m \leq M_P(N_P^m + B) - \theta N_P^{m*}(x_P; M_P) - K(x_P; M_P),$$

$$C^o \leq F(A, N_P^m) - M_P(N_P^m + B),$$

$$B \geq N_P^{mb}$$

Note that:

- for $A = K_P > 0$, $F(A) = U_P^o(K_P, N_P^m, N_P^o; M_P)$.

- $(K_P^*(x_P; M_P), N_P^{m*}(x_P; M_P)) \in \Gamma_P(A + h, N_P^m, N_P^o, M_P),$

- $(K_P^*(x_P; M_P), N_P^{m*}(x_P; M_P)) \in \Gamma_P(A - h, N_P^m, N_P^o, M_P)$ for $h$ small enough

(since, at an optimum, $C^o > 0$ because $\frac{\partial u_D(C^o, N_P^o)}{\partial C^o}|_{C^o=0} = \infty$),

- $G(A + \varepsilon) \leq U_P^o(K_P + \varepsilon, N_P^m, N_P^o; M_P)$, \(\forall \varepsilon \in [-h, h]\),

- since $u_D$ is strictly concave and $\Psi_D$ weakly concave, $F$ is strictly concave,

- since $u_D$ and $\Psi_D$ are differentiable, $F$ is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

$$\frac{\partial U_P^o(x_P; M_P)}{\partial K_P} = G'(A) = \beta \frac{\partial u_D(C^o, N_P^o)}{\partial C^o} > 0 \quad (T.2)$$
Define

\[ G(A) \equiv \max_{C^m, C^o, B} \{ \beta u_D(C^o, A) + \Psi_D[N^m_P, U(C^m, N^m_P) + U^*_P(K'_p(x; M_P), N^{m*}_p(x; M_P), N^m_P)] \} \]

subject to

\[ C^m \leq M_P(N^m_P + B) - \theta N^{m*}_P(x; M_P) - K'(x; M_P), \]
\[ C^o \leq F(K_P, N^m_P) - M_P(N^m_P + B), \]
\[ B \geq N^{m*}_P \]

Note that:

- for \( A = N^o_P > 0 \), \( G(A) = U^*_P(K_P, N^m_P, N^o_P; M_P) \),
- since \( \Gamma_P \) is independent on \( N^o_P \),
  \( (K'_p(x; M_P), N^{m*}_P(x; M_P)) \in \Gamma_P(K_P, N^m_P, A + h; M_P) \),
  \( (K'_p(x; M_P), N^{m*}_P(x; M_P)) \in \Gamma_P(K_P, N^m_P, A - h; M_P) \),
- \( G(A + \varepsilon) \leq U^*_P(K_P, N^m_P, N^o_P + \varepsilon; M_P), \forall \varepsilon \in [-h, h] \),
- since \( u_D \) is strictly concave, \( G \) is strictly concave,
- since \( u_D \) is differentiable, \( G \) is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

\[ \frac{\partial U^*_P(x; M_P)}{\partial N^o_P} = G'(A) = \beta \frac{\partial u_D(C^o, N^o_P)}{\partial N^o_P} > 0. \]  \hspace{1cm} (T.3)

**T.4.3** \( U^*_P(K_P, N^m_P, N^o_P; M_P) \) differentiable in \( N^m_P > 0 \)

Define

\[ G(A) \equiv \max_{C^m, C^o, B} \{ \beta u_D(C^o, N^o_P) + \Psi_D[A/M_P, u_D(C^m, A/M_P) + U^*_P(K'_p(x; M_P), N^{m*}_p(x; M_P), A/M_P)] \} \]

subject to

\[ C^m \leq A + M_P B - \theta N^{m*}_P(x; M_P) - K'(x; M_P), \]
\[ C^o \leq F(K_P, A/M_P) - (A + M_P B), \]
\[ M_P B \geq A.b. \]

**Case (i):** \( M_P B > A.b. \)

Note that:
• for \( A = M_P N_P^m > 0 \), \( G(A) = U_P^*(K_P, N_P^m, N_P^o; M_P) \).

• for \( h \) small enough, \( M_P B > (A + h) \hat{b} \) and

\[
(K_P(x_P, M_P), N_P^{m*}(x_P, M_P)) \in \Gamma_p(K_P, (A + h)/M_P, N_P^o; M_P),
\]

(since, at an optimum, \( C^o > 0 \) because \( \frac{\partial u_D(C^o, N^o)}{\partial C^o} \mid_{C^o=0} = \infty \)),

• for \( h \) small enough, \( M_P B > (A + h) \hat{b} \) and

\[
(K_P'(x_P, M_P), N_P^{m*}(x_P, M_P)) \in \Gamma_p(K_P, (A + h)/M_P, N_P^o; M_P),
\]

(since, at an optimum, \( C^o > 0 \) because \( \frac{\partial u_D(C^o, N^o)}{\partial C^o} \mid_{C^o=0} = \infty \)),

• \( G(A + \varepsilon) \leq U_P^*(K_P, N_P^m + \varepsilon/M_P, N_P^o; M_P), \forall \varepsilon \in [-h, h] \),

• since \( u_D \) is strictly concave and \( \Psi_D \) weakly concave, \( G \) is strictly concave,

• since \( u_D \) and \( \Psi_D \) are differentiable in \( N^m \) and \( U_P^* \) differentiable in \( N^o \), \( G \) is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

\[
\frac{\partial U_P^*(x_P; M_P)}{\partial N_P^m} = G'(A)(dA/dN_P^m) = G'(A)M_P \tag{4.10}
\]

\[
= \beta \frac{\partial u_D(C^o, N^o)}{\partial C^o}(F_N(K_P, N_P^m) - M_P)
\]

\[+ \frac{\partial \Psi_D}{\partial N^m} + \frac{\partial \Psi_D}{\partial U}[M_P \frac{\partial u_D(C^m, N^m_P)}{\partial C^o} + \frac{\partial u_D(C^m, N^m_P)}{\partial N^m} + \beta \frac{\partial u_D(C^o', N^o'_P)}{\partial N^o}].
\]

Or, since \( F_N(K_P, N_P^m) = M_P \)

\[
\frac{\partial U_P^*(x_P; M_P)}{\partial N_P^m} = G'(A)(dA/dN_P^m) = G'(A)M_P \tag{4.11}
\]

\[= \frac{\partial \Psi_D}{\partial N^m} + \frac{\partial \Psi_D}{\partial U}[M_P \frac{\partial u_D(C^m, N_P^m)}{\partial C^o} + \frac{\partial u_D(C^m, N_P^m)}{\partial N^m} + \beta \frac{\partial u_D(C^o', N^o'_P)}{\partial N^o}] > 0.
\]

**Case (ii):** \( M_P B = A \hat{b} \) and \( \lambda_B > 0 \).

Note that:

• for \( A = M_P N_P^m > 0 \), \( F(A) = U_P^*(K_P, N_P^m, N_P^o; \hat{x}) \).

• for \( h \) small enough, \( M_P B = (A + h) \hat{b} \), \( \lambda_B > 0 \) and

\[
(K_P(x_P, \hat{x}), N_P^{m*}(x_P; M_P)) \in \Gamma_p(K_P, (A + h)/M_P, N_P^o; M_P),
\]

(since, at an optimum, \( C^o > 0 \) because \( \frac{\partial u_D(C^o, N^o)}{\partial C^o} \mid_{C^o=0} = \infty \)),

T-14
• for $h$ small enough, $(K_D, (A - h)/w(\hat{x}), N_D^o \in A_{CO}$ (since $A_{CO}$ open) and

$\left( K_D(x_D, \hat{x}), N_D^{m*}(x_D, \hat{x}) \right) \in \Gamma_D(K_D, (A - h)/w(\hat{x}), N_D^o; \hat{x})$,

(since, at an optimum, $C^m > 0$ because $\frac{\partial u_D(C^m, N^m)}{\partial C^m}|_{C^m=0} = \infty$),

$\bullet$ $G(A + \varepsilon) \leq U^*_P(K_P, N^m_P + \varepsilon/M_P, N^o_P; M_P), \forall \varepsilon \in [-h, h]$,

$\bullet$ since $u_D$ is strictly concave and $\Psi_D$ weakly concave, $G$ is strictly concave,

$\bullet$ since $u_D$ and $\Psi_D$ are differentiable in $N^m$ and $U^*_P$ differentiable in $N^o$, $G$ is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

$$\frac{\partial U^*_P(x_P; M_P)}{\partial N^m_P} = G'(A)(dA/dN^m_P) = G'(A)M_P$$

(T.6)

$$= \beta u_D(C^o, N^o_P)(F_N(K_P, N^m_P) - M_P(1 + b) + \frac{\partial \Psi_D}{\partial N^m}$$

$$+ \frac{\partial \Psi_D}{\partial U}[M_P\frac{\partial u_D(C^m, N^m_P)}{\partial C^m}(1 + b) + \frac{\partial u_D(C^m, N^m_P)}{\partial N^m} + \beta \frac{\partial u_D(C^o, N^o_P)}{\partial N^o}]$$

Or, since $M_P = F_N(K_P, N^m_P)$

$$\frac{\partial U^*_P(x_P; M_P)}{\partial N^m_P} = \left[ \frac{\partial \Psi_D}{\partial U} \frac{\partial u_D(C^m, N^m_P)}{\partial C^m} - \beta \frac{\partial u_D(C^o, N^o_P)}{\partial C^o} \right] M_Pb$$

(T.7)

$$+ \frac{\partial \Psi_D}{\partial N^m} + \frac{\partial \Psi_D}{\partial U}[M_P\frac{\partial u_D(C^m, N^m_P)}{\partial C^m} + \frac{\partial u_D(C^m, N^m_P)}{\partial N^m} + \beta \frac{\partial u_D(C^o, N^o_P)}{\partial N^o}] > 0$$

[Note that the first term is $-\lambda_B b$ and the rest is the same as in case (i).]

**Case (iii):** $M_P B = \Lambda \underline{b}$ and $\lambda_B = 0$.

Note that:

• for $A = M_P N^m_P > 0$, $G(A) = U^*_P(K_P, N^m_P, N^o_P; M_P)$.

• for and $h > 0$, $M_P B = (A + h)\underline{b}$ and $\lambda_B > 0$, i.e. we are in case (ii),

• for and $h < 0$, $M_P B = (A - h)\underline{b}$ i.e. we are in case (i),

• $G(A + \varepsilon) \leq U^*_P(K_P, N^m_P + \varepsilon/M_P, N^o_P; M_P), \forall \varepsilon \in [-h, h]$,

Hence, we need to show that

$$\lim_{h>0, h \to 0} \frac{\partial U^*_P(x_P; M_P)}{\partial N^m_P} |_{ii} = \lim_{h<0, h \to 0} \frac{\partial U^*_P(x_P; M_P)}{\partial N^m_P} |_{i}$$

T-15
Note that 

\[ \lim_{h \to 0, h > 0} \frac{\partial U_P^*(x_P; M_P)}{\partial N_P^m} |_{ii} = - \lim_{h \to 0, h > 0} \lambda_B b + \frac{\partial U_P^*(x_P; M_P)}{\partial N_P^m} |_{i}. \]

Since, \( \lambda_B \to 0 \) as \( h \to 0 \), we have \( \frac{\partial U_P^*(x_P; M_P)}{\partial N_P^m} |_{iii} = \frac{\partial U_P^*(x_P; M_P)}{\partial N_P^m} |_{i}. \)

**T.5 Detailed proof of Proposition S.3**

**Proposition:**

\( U_P^* \) is differentiable with respect to the parameter \( b \) and the Pseudo Planner’s policy functions are continuous in \( b \). Therefore, household policy functions and equilibrium prices are continuous in \( b \).

**Proof.**

Let’s add \( b \) as a state variable with law of motion \( b' = b \).

We want to show that \( U_P^*(x_P; b, M_P) \) differentiable in \( b \).

To do so, define

\[
F(A) \equiv \max_{C^m, C^o, B} \{ \beta U(C^o, N_P^m) + \Psi_D[N_P^m, U(C^m, N_P^m) + U_P^*(x_P, b)] \}
\]

subject to

\[
C^m \leq M_P(N^m_P + B) - \theta N^m_P(x_P) - K^*(x_P),
C^o + M_P(N^m_P + B) \leq F(K^*_P, N^m_P),
M_P B \geq A,
\]

Define \( b^* \in [-1, \bar{b}] \) such that, given \((K^*_P, N^m_P, N^o_P),\)

\[
\beta \frac{\partial u_D(o, x_P, b^*)}{\partial C^o} = \frac{\partial \Psi_D(N^m_P, u_D(m, x_P, b^*))}{\partial U} + U_P^*(x_P, b^*) \frac{\partial u_D(m, x_P, b^*)}{\partial C^m}
\]

and define \( B^*(x_P, F_N(x_P)) = N^m_P b^* \).

That is, the constraint is just binding. \( b^* \) exists if \( \partial \Psi_D/\partial U > 0 \). If not, only case ii is relevant.

i. Case 1: \( b < b^* \).

Note that:

- for \( A = M_P N^m_P b \) \( F(A) = U_P^*(x_D, b, M_P) \).
- for \( h \) small enough, \( x_P \) s.t. \( b < b^* \) still holds for \( (A + h) \)
- \((K^*_P(x_P, b, M_P), N^m_P(x_P, b, M_P)) \) feasible for \( (A + h) \)
  (since intertemporal constraint is independent on \( b \))
- \( F(A + \varepsilon) \leq U_P^*(x_P, (A + \varepsilon)/(M_P N^m_P); M_P), \forall \varepsilon \in [-h, h] \)

T-16
• Since \( u_D \) is strictly concave and \( \Psi_D \) weakly concave, \( F \) is strictly concave.

• Since \( u_D \) and \( \Psi_D \) are differentiable in \( C \), \( F \) is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

\[
\frac{\partial U^*_P(x_P, b_P, M_P)}{\partial b} \bigg|_{case1} = F'(A)(dA/db) = 0
\]

ii. Case 2: \( b > b^* \).

Note that:

• for \( A = M_P N_P b_P \) \( F(A) = U^*_P(x_D, b_P, M_P) \).

• for \( h \) small enough, \( x_P \) s.t. \( b > b^* \) still holds for \( (A \pm h) \)

• \((K_P^*(x_P, b_P, M_P), N_P^m(x_P, b_P, M_P))\) feasible for \( (A \pm h) \)

  (since intertemporal constraint is independent on \( (b) \))

• \( F(A + \varepsilon) \leq U^*_P(x_P, (A + \varepsilon)/(M_P N_P); M_P), \forall \varepsilon \in [-h, h] \)

• Since \( u_D \) is strictly concave and \( \Psi_D \) weakly concave, \( F \) is strictly concave.

• Since \( u_D \) and \( \Psi_D \) are differentiable in \( C \), \( F \) is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

\[
\frac{\partial U^*_P(x_P, b_P, M_P)}{\partial b} \bigg|_{case2} = F'(A)(dA/db) = F'(A)M_P N_P^m \tag{T.8}
\]

\[
= -\beta \frac{\partial u_D(C^0(x_P, x_P^*(x_P)), N_P^o)}{\partial C^0} M_P N_P^m + \frac{\partial \Psi_D}{\partial U} \frac{\partial u_D(C^m(x_P, x_P^*(x_P)), N_P^o)}{\partial C^m} M_P N_P^m
\]

\[
= M_P N_P^m \lambda_B
\]

iii. Case 3: \( b = b^* \), i.e. \( x_P \in A_{UC_P} \).

Note that:

• for \( A = M_P N_P b_P \) \( F(A) = U^*_P(x_D, b_P; M_P) \).

• for and \( h > 0 \), we are back to case 1 for \( (A + h) \) and

T-17
• for and $h < 0$, we are back to case 2 for $(A + h)$.

• $F(A + \varepsilon) \leq U^*_p(x_p, (A + \varepsilon)/(M_pN^*_p); M_p), \forall \varepsilon \in [-h, h]$  

Hence, we need to show that

$$\lim_{h>0, h \to 0} \frac{\partial U^*_p(x_p, b, M_p)}{\partial b}_{|\text{case 1}} = \lim_{h<0, h \to 0} \frac{\partial U^*_p(x_p, b, M_p)}{\partial b}_{|\text{case 2}}$$

Since, $\lambda_B(x_p, x^*_p(x_p), M_p) \to 0$ as $h \to 0$, we have $\frac{\partial U^*_p(x_p, b, M_p)}{\partial b}_{|\text{case 3}} = 0$

Hence, $U^*_p$ is differentiable in $b$. Thus $U^*_p$ is continuous in $b$ and by the Theorem of the Maximum, the optimal policy functions, $x^*_p$ are continuous in $b$. Therefore, using the equivalence in Proposition S.5 equilibrium prices, $w(\tilde{x}'(b)) = F_N(x^*_p(b))$ and $r(\tilde{x}'(b)) = F_K(x^*_p(b))$, are continuous in $b$.  

\[ \square \]